Problem 1

For $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$, we define[a]

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}.$$

In this problem, we will deduce that for $|x| < 1$,

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n. \tag{1}$$

1. Show that $\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ converges if $|x| < 1$ using the ratio test.
2. Let $f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$ for $|x| < 1$. Show that $(1+x)f'(x) = \alpha f(x)$.
3. Suppose $f : (-1, 1) \to \mathbb{R}$ is a differentiable function satisfying $(1+x)f'(x) = \alpha f(x)$. Show that $f(x) = c(1+x)^\alpha$ for some constant c.

   Hint: Consider $g(x) = f(x)/(1+x)^\alpha$.
4. Conclude (1), i.e. $c = 1$ in the previous subproblem.

[a] This extends the definition of the binomial $\binom{m}{n}$ for $m \in \mathbb{N}, 0 \le n \le m$.

Problem 2

Suppose that $(f_n)_{n\in\mathbb{N}} : [a, b] \to \mathbb{R}$ is a sequence of continuous functions which converges uniformly to f. Show that if $x_n \to x$ where $x_n \in [a, b]$, then $f_n(x_n) \to f(x)$. Is this true if we don't assume the f_n are continuous? Is it true if the convergence is not uniform?

Problem 3

Suppose that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is an even function. Show that $a_n = 0$ for every odd $n \in \mathbb{N}$. If f is odd instead, show that $a_n = 0$ for every even $n \in \mathbb{N}$.